Matrix Algebra

Matrices

A *matrix* is a rectangular array of elements arranged in rows and columns, e.g.:

$$\mathbf{A} = \begin{bmatrix} 6 & 13 \\ 9 & 21 \\ 12 & 5 \end{bmatrix}, \text{ where } \mathbf{A} \text{ here is a 3 row by 2 column matrix.}$$

More generally, a matrix of this size or dimension (3 by 2) can be written as

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}. \text{ where the } a_{ij}\text{'s are the elements of the matrix.}$$

Even more generally, the $(r \times c)$ matrix **A** can be written as:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1c} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2c} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{ic} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{r1} & a_{r2} & \cdots & a_{rj} & \cdots & a_{rc} \end{bmatrix}$$

where $\mathbf{A} = \left[a_{ij} \right];\ i = 1,\ldots,r;\ j = 1,\ldots,c;$ and *r* is the number of rows and *c* the number of columns in the matrix.

Special Matrices

1) the *square matrix* ($r = c$), e.g.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix};$$

2) the 1-d *row vector* (an $r \times 1$ matrix), e.g.

$$\mathbf{a} = \begin{bmatrix} a_{11} \\ a_{21} \\ \cdots \\ a_{r1} \end{bmatrix};$$

3) the 1-d *column vector* (a $1 \times c$ matrix), e.g.

$$\mathbf{b} = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1c} \end{bmatrix};$$

4) a *scalar*, or a 1×1 matrix: $\mathbf{a} = \begin{bmatrix} a_{11} \end{bmatrix}$

5) the *zero matrix*:

$$\mathbf{0} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{bmatrix};$$

6) an $r \times r$ *diagonal matrix*:

$$\mathbf{T} = \begin{bmatrix} t_1 & 0 & \cdots & 0 & 0 \\ 0 & t_2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & t_{r-1} & 0 \\ 0 & 0 & \cdots & 0 & t_r \end{bmatrix};$$

7) the *identity matrix*:

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & 1 \end{bmatrix};$$ (a square matrix of zeros, with ones along the diagonal), and

8) a *vector of ones*: $\mathbf{1} = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}$.

Matrix operations and further definitions:

1) *transposition*, e.g., if

$$\mathbf{A} = \begin{bmatrix} 1 & 6 \\ 5 & 3 \\ 0 & 2 \end{bmatrix},$$ then $\mathbf{A}^T$ or $\mathbf{A}' = \begin{bmatrix} 1 & 5 & 0 \\ 6 & 3 & 2 \end{bmatrix}$, or if

$$\mathbf{C} = \begin{bmatrix} 7 \\ 9 \\ 5 \end{bmatrix},$$ then $\mathbf{C}' = \begin{bmatrix} 7 & 9 & 5 \end{bmatrix}$, or more generally, if $\mathbf{A}$ (an $r \times c$ matrix) is

$$\mathbf{A} = \begin{bmatrix} a_{11} & \cdots & a_{1c} \\ \cdots & \cdots & \cdots \\ a_{r1} & \cdots & a_{rc} \end{bmatrix}, \text{ then } \mathbf{A}' = \begin{bmatrix} a_{11} & \cdots & a_{r1} \\ \cdots & \cdots & \cdots \\ a_{1c} & \cdots & a_{rc} \end{bmatrix}.$$

2) a matrix can be said to be *symmetric* if $\mathbf{A} = \mathbf{A}'$;

3) *equality*: $\mathbf{A} = \mathbf{B}$, if $a_{ij} = b_{ij}$, all i and j.

4) *addition* and *subtraction* (of corresponding elements): If

$$\mathbf{A} = \begin{bmatrix} 1 & 4 \\ 2 & 3 \\ 3 & 6 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 4 \end{bmatrix}, \text{ then}$$

$$\mathbf{A}+\mathbf{B} = \begin{bmatrix} 1+1 & 4+2 \\ 2+2 & 5+3 \\ 3+3 & 6+4 \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ 4 & 8 \\ 6 & 10 \end{bmatrix}, \text{ and } \mathbf{A}-\mathbf{B} = \begin{bmatrix} 1-1 & 4-2 \\ 2-2 & 5-3 \\ 3-3 & 6-4 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 2 \\ 0 & 2 \end{bmatrix}.$$

More generally, $\mathbf{C} = \mathbf{A}+\mathbf{B}$, where $c_{ij} = a_{ij} + b_{ij}$.

In order to be added or subtracted, the two matrices must be *conformable*. In the case of addition or subtraction, they must be of the same shape. If **A** is a p row by q column matrix, which can be written as ${}_p\mathbf{A}_q$, then **B** must also be a p row by q column matrix, and the resultant matrix **C** will also be a p row by q column matrix, or ${}_p\mathbf{C}_q = {}_p\mathbf{A}_\mathrm{q} + {}_p\mathbf{B}_q$.

5) *scalar multiplication* (multiplication of each element by the same scalar value): if

$$\mathbf{A} = \begin{bmatrix} 2 & 7 \\ 9 & 3 \end{bmatrix}, \text{ then } 4 \times \mathbf{A} = 4 \cdot \begin{bmatrix} 2 & 7 \\ 9 & 3 \end{bmatrix} = \begin{bmatrix} 8 & 28 \\ 36 & 12 \end{bmatrix}.$$

6) *matrix multiplication*: if

$$\mathbf{A} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}, \text{ and } \mathbf{B} = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}, \text{ then}$$

$$\mathbf{C} = \mathbf{AB} = \begin{bmatrix} (1\cdot 2 + 3\cdot 0) & (1\cdot 3 + 3\cdot 4) \\ (2\cdot 2 + 4\cdot 0) & (3\cdot 3 + 4\cdot 4) \end{bmatrix}.$$

Note that matrix multiplication requires that the matrices be *conformable* for multiplication, e.g. if **A** is a p row by q column matrix, and **B** is a q row by r column matrix then ${}_p\mathbf{A}_q$ and ${}_q\mathbf{B}_r$ can be said to be conformable (because **A** has q columns, and **B** has q rows), and their product **C** can be obtained with dimensions of p rows by r columns, or ${}_p\mathbf{C}_r = {}_p\mathbf{A}_q\mathbf{B}_r$.

Note that **A** postmultiplied by **B** is not conformal, except if $p = r$.

Matrix multiplication can also be considered as a set of sums of cross-products, one for each element of the product matrix, or

$$ {}_p\mathbf{C}_r = {}_p\mathbf{A}_q\mathbf{B}_r, \text{ where } c_{ij} = \sum_{k=1}^{q} a_{ik}b_{jk}. $$

7) matrix *inversion*: For a square matrix **A**, the inverse of **A** is the matrix that when premultiplied or postmultiplied by **A** yields the identity matix **I,** or

$$ \mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}. $$

Matrix inversion is analogous in some ways to scalar division:

If $(1/x)$ is the inverse of x, then $x \cdot \frac{1}{x} = xx^{-1} = x^{-1}x = 1.$

8) *Linear combinations* can be compactly written in matix terms. If

$$ \begin{aligned} z_1 &= a_1X_{11} + a_2X_{12} + \ldots + a_pX_{1p} \\ z_2 &= a_1X_{21} + a_2X_{22} + \ldots + a_pX_{2p} \\ &\ldots \\ z_i &= a_1X_{i1} + a_2X_{i2} + \ldots + a_pX_{ip} \\ &\ldots \\ z_n &= a_1X_{n1} + a_2X_{n2} + \ldots + a_pX_{np} \end{aligned} $$

then this system of equations can be written in more compact form as

$$ \underset{(N\times 1)}{\mathbf{z}} = \underset{(N\times p)}{\mathbf{X}}\ \underset{(p\times 1)}{\mathbf{b}}, \text{ where} $$

$$ \underset{(N\times 1)}{\mathbf{z}} = \begin{bmatrix} z_1 \\ z_2 \\ \cdots \\ z_N \end{bmatrix}, \ \underset{(p\times 1)}{\mathbf{a}} = \begin{bmatrix} a_0 \\ a_1 \\ \cdots \\ a_p \end{bmatrix}, \text{ and } \underset{(N\times p)}{\mathbf{X}} = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1p} \\ x_{21} & x_{22} & \cdots & x_{2p} \\ \cdots & \cdots & \cdots & \cdots \\ x_{N1} & x_{N2} & \cdots & x_{Np} \end{bmatrix}. $$

9) *Eigenvalues* and *eigenvectors*. If **R** is a $(p \times p)$ square matrix, then there is another $(p \times p)$ square matrix **E**, such that

$\mathbf{RE} = \mathbf{EV}$, where **V** is a $(p \times p)$ diagonal matrix.

The first column of these matrices can be written as $\mathbf{Re}_1 = \lambda_1\mathbf{e}_1$, or $(\mathbf{R} - \lambda_1\mathbf{I})\mathbf{e}_1 = \mathbf{0}$.

10) Matrix *quadratic forms*: If **A** is an $(n \times n)$ and **x** is an $(n \times 1)$ column vector, then

$$ Q = \mathbf{x}'\mathbf{A}\mathbf{x} = \mathbf{x}\mathbf{A}\mathbf{x}' = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i a_{ij} x_j. $$